MID-SEMESTER EXAMINATION
M. MATH II YEAR, II SEMESTER 2016-2017
ERGODIC THEORY

Max. Score: 100 Time limit: 3hrs.

1. Let $(\Omega, \mathcal{F}, P, T)$ be a dynamical system and $A \in \mathcal{F}$ with $P(A) > 0$. Prove that there exists a positive integer n with $P(T^{-n}A \cap A) > 0$. [10]

2. Let $\Omega = S^1, \mathcal{F} =$ Borel σ - field, $P =$ normalized Haar measure $, Tz = cz$ where $c \in S^1$ is not a root of unity. Find all eigen values and eigen functions of T. [20]

3. Define $T : [0, \infty) \to [0, \infty)$ by $Tx = x^2$. Find all Borel probability mesures P on $[0, \infty)$ which are absolutely continuous w.r.t. Lebesgue measure and which are invariant for T. [20]

4. Let $(\Omega, \mathcal{F}, P, T)$ be a dynamical system and T be ergodic. Let f be a non-negative measurable function on $(\Omega, \mathcal{F})$ such that $\limsup\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=0}^{n-1} f(T^k\omega) < \infty$ almost surely. Show that $f \in L^1(P)$. [15]

5. Let $\{T_t : t \geq 0\}$ be a flow on $(\Omega, \mathcal{F}, P)$. Show that $n \int\limits_1^{1+\frac{1}{n}} f(T_t)dt \to f(T_1)$ almost surely as $n \to \infty$. [15]

6. Let $P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Let T be the shift associated with the Markov chain with transition matrix P. Is T ergodic? If so, is it mixing. If so what is $\lim\limits_{n\to\infty} P^n$? [20]